HUB-EP-97/22
MS-TPI-97-1

The Interface Tension of the 3-Dimensional Ising Model in the Scaling Region

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Abstract

Using the Monte Carlo method, we determine the free energy of the interface of the 3D Ising model in the scaling region. By integrating the interface energies over the inverse temperature β, we obtain estimates for the free energies of interfaces with cross sections up to 96 by 96, and for a range $0.223 \leq \beta \leq 0.23$. Our data yield a precise estimation of the interface tensions σ. We determine the amplitude σ_0 in the critical law $\sigma \sim \sigma_0\, t^{\mu}$ and estimate the combination $\sigma\xi^2$ which yields the universal constant R_- in the critical limit.

1 Introduction

The numerical determination of interface free energies and interface tensions in statistical models is not straightforward. Part of the difficulties stem from the fact that the interface free energy is the logarithm of a ratio of partition functions rather than an expectation value. This prohibits a straightforward application of the Monte Carlo method. Various methods have been invented to overcome this difficulty, each of which has its own merits and disadvantages. For a number of numerical studies of the 3D Ising interfaces, all done in the early nineties, see, e.g., [1, 2, 3, 4, 5, 6, 7].

In this paper, we pick up our methods from [3], focusing on the critical region of the model. We present data for the free energies and interface tensions of interfaces in the 3D Ising model. The basic idea, which is quite old by now, see e.g. [8], is to compute the derivative of the interface free energy with respect to the inverse temperature β, i.e., the interface energy. The interface energy is the difference between the energies of the system with periodic and with antiperiodic boundary conditions (in one of the lattice directions), respectively. The interface free energy is then obtained by (numerical) integration over the energy estimates.

A major advantage of the integration method is that it allows to study interfaces of quite large cross sections. This is necessary if one wants to get systematic errors in the estimates of the interface tension under control, especially in the scaling region.

In the present paper, we extend our previous results [3, 7] in several ways:

- We combine the integration method with the boundary flip method [5]. The latter serves to provide the initial values for the integration.
- We include new efficient algorithms in the simulation of the systems with antiperiodic and periodic boundary conditions.
- We increase significantly the statistics of the simulations.
- We include larger cross sections of the interfaces in the analysis.
- Due to the better precision of our data we can do more refined fits to theoretical predictions for the quantities in question.

This article is organized as follows: In Section 2 we introduce the model, notation and setup. Furthermore, we discuss the essentials of our method to compute interface properties and also discuss finite size effects. Section 3 is devoted to a short description of our Monte Carlo procedures. In section 4, we explain how we determine the interface free energy estimates. These are the basis for the estimation of the interface tensions by various fitting procedures. This is discussed in section 5. Section 6 reports on our fits to determine the amplitude σ_0 and other parameters in the critical law. It follows section 7, where we estimate the universal amplitude ratio R_{-}. We close with a short summary.

2 Interfaces in the Ising Model

Consider the 3D Ising model on the simple cubic lattice, with the Hamiltonian

$$H = -\sum_{<x,y>} s_x s_y \,, \qquad s_x = \pm 1 \,. \tag{1}$$

The sites of the lattice are labelled by integer coordinates $x = (x_1, x_2, x_3)$. The sum in eq. (1) is over all (unordered) nearest neighbour pairs of sites in the lattice. The partition function is

$$Z = \sum_{\{s\}} \exp\left(-\beta H\right) . \tag{2}$$

Here, the summation is over all possible configurations of the Ising spins. The pair interaction is normalized such that $\beta = 1/(k_B T)$, where k_B denotes Boltzmann's constant, and T is the temperature.

At a critical coupling β_c (the estimate of a recent study [9] is $\beta_c = 0.2216544(6)$) the model undergoes a second order phase transition. For $\beta > \beta_c$, the system shows spontaneous breaking of the reflection symmetry.

In order to study interfaces separating extended domains of different magnetization, we consider lattices with extension $L_1 = L_2 = L$ in the x_1- and x_2-directions and with extension L_3 in the x_3-direction. We generalize eq. (1) to

$$H = -\sum_{<x,y>} k_{xy}\, s_x s_y \, . \tag{3}$$

The lattice becomes a torus by regarding the opposite boundary planes as neighbour planes. In addition to periodic boundary conditions, where $k_{xy} = 1$ for all links, we consider so called antiperiodic boundary conditions in x_3-direction. Antiperiodic boundary conditions in x_3-direction are imposed by $k_{xy} = -1$ for the links that connect the uppermost with the lowermost plane. For the other links we keep $k_{xy} = 1$. In the following we will indicate the type of boundary condition by the subscript p for periodic and a for antiperiodic. In particular, the Hamiltonian with periodic boundary conditions is denoted by H_p, and the Hamiltonian with antiperiodic boundary conditions by H_a.

For sufficiently large β and large enough L, the imposure of antiperiodic boundary conditions forces the system to develop exactly one interface, a region where the magnetization rapidly changes sign. This interface is parallel to a (001) lattice plane.

Let us mention that the Ising (001) interface undergoes a *roughening transition* at an inverse temperature β_R that is nearly twice as large as the bulk transition coupling β_c. The presently most accurate estimate of the inverse roughening temperature is $\beta_R = 0.40758(1)$ [10]. In the region of $\beta_c < \beta < \beta_R$ the interface is rough. It is smooth (rigid) for $\beta > \beta_R$.

2.1 Definition of the Interface Free Energy

We define the interface free energy as the difference of the free energy of a system with antiperiodic boundary conditions, $F_a = -\ln Z_a$, and the free energy of a system of the same size but periodic boundary conditions, $F_p = -\ln Z_p$, viz.

$$F_s = F_a - F_p + \ln L_3 \, . \tag{4}$$

The term $\ln L_3$ takes care of the possible translations of the interface along the 3-direction. For a more detailed discussion of this definition, see refs. [3, 5, 7, 11].

2.2 Finite Size Behaviour of the Interface Free Energy

A detailed discussion of the dependence of F_s on L_3 can be found in ref. [7], section 2. One finds that for

$$\xi_b \ll L_3 \ll \xi_t \,, \tag{5}$$

F_s is essentially independent of L_3. Here, ξ_b is the bulk correlation length, and ξ_t denotes the tunneling correlation length.

In ref. [7] we found convergence of the interface energy within our numerical accuracy for about $L_3 > 15\ \xi_b$. The convergence was found for a large range of interface extensions L. Since $\xi_t = \frac{1}{2}\exp(F_s)$, it is usually no problem to fulfill the second inequality of eq. (5).

The dependence of F_s on L can be discussed in the framework of effective interface models. It is in a natural way related to the question of how to define and determine the interface tension σ. Note that σ can unambiguously be defined through

$$\sigma_\infty = \lim_{L\to\infty} \frac{F_s}{L^2} \,. \tag{6}$$

In order to extract reliable estimates of σ_∞ from finite L data (cf. section 5 below), one profits very much from a more detailed information on the finite L behaviour of F_s.

It was observed already in [3] that for sufficiently large L, F_s follows with good precision

$$F_s \simeq C_s + \sigma\, L^2 \,, \tag{7}$$

where C_s is a constant. This type of behaviour is suggested by the 1-loop approximation of the capillary wave model [12, 13, 11] and also by the 1-loop semiclassical expansion in the field theoretic framework [14].

In [11] it was derived from the 2-loop-expansion of the capillary wave model that the free energy should behave as

$$F_s \simeq C_s + \sigma\, L^2 - \frac{1}{4\sigma L^2} \,. \tag{8}$$

2.3 How to Compute F_s

In this subsection we outline our strategy to compute the interface free energy. The boundary flip algorithm allows for a direct Monte Carlo measurement of F_s. However, the method works well only for $F_s < 10$. Therefore we combined the boundary flip method with the integration method first used in ref. [8]. Taking the derivative of F_s with respect to β, we obtain the interface energy E_s,

$$E_s = E_a - E_s \ , \tag{9}$$

where E_a and E_p are the expectation values of the corresponding Hamiltonians, H_a and H_p. E_a and E_p can be computed by Monte Carlo simulations of systems with antiperiodic and periodic boundary conditions, respectively. The interface free energy in a range of inverse temperatures is then obtained by numerically performing the integration

$$F_s(\beta) = F_s(\beta_0) + \int_{\beta_0}^{\beta} d\beta'\, E_s(\beta') \ . \tag{10}$$

Note that one could in principle choose any β_0 as the starting point of the integration. We chose β_0 close to criticality, so that the condition $F_s < 10$ is fulfilled also for the larger interface cross sections (cf. table 1 and section 4 below).

3 The Monte Carlo Algorithms

For the simulations we used four types of algorithms:

- boundary flip algorithm
- single cluster algorithm
- surface cluster algorithm
- demon algorithm

The boundary flip algorithm was used to determine the interface free energy at the smallest inverse temperature β considered for a given lattice size. For the measurement of the energy with periodic and antiperiodic boundary conditions we simulated the system with the single cluster algorithm [15] (combined with a surface cluster algorithm [16] in case of antiperiodic boundary conditions) for a β-range closest to the critical value, while the demon algorithm was used for the remaining β-values up to $\beta = 0.23$.

The boundary flip algorithm [5] allows to simulate an ensemble that contains periodic as well as antiperiodic boundary conditions. The partition function of this system is given by $Z = Z_p + Z_a$. The ratio Z_a/Z_p is given by $< \delta_{b,a} > / < \delta_{b,p} >$, where $\delta_{b,a}$ is equal to one if the boundary conditions are antiperiodic, and zero else. $\delta_{b,p}$ is defined analogously. The boundary flip algorithm is a modification of the cluster algorithm. For a detailed discussion of the algorithm see ref. [18].

The single cluster algorithm was introduced in [15] as an improvement of the multi-cluster algorithm of Swendsen and Wang [19]. For a detailed discussion of its critical dynamical behaviour see [17].

In case of the antiperiodic boundary conditions, and thus the existence of an interface, a straightforward application of the bulk cluster algorithm is not appropriate. The interface is correlated on all length scales, whereas the bulk correlation length is finite. Furthermore, competing interactions are induced by the interface, and cluster algorithms become inefficient for frustrated systems. We therefore combined the single cluster algorithm with the interface cluster proposed in ref. [16], and a slight modification of it which is more suitable for simulations close to the bulk critical temperature.

For the larger β-values we used a demon algorithm [20] combined with canonical updates of the demons [21]. The algorithm is a local algorithm. Its advantage is that compared to standard algorithms one needs much less random numbers.

The algorithm can easily be implemented in multi-spin coding fashion. For each spin (and demon) only one bit is used, and the operations are done simultaneously on each bit of a given word (in our case 64 bits per word). For a detailed discussion of the algorithm see ref. [7].

In order to give an idea of the CPU-times required we give some typical update times for a DEC Alpha 250 4/266 workstation, where most of our simulations were performed. One boundary flip update for the 96^3 lattice at $\beta = 0.2219$ takes 0.515 sec. The update of a single site with the single cluster algorithm takes 1.9×10^{-6} sec. This time should be compared with the performance of the demon program in multispin coding: Here the update of a single spin takes 46×10^{-9} seconds on an HP 735 and 21×10^{-9} seconds on a DEC Alpha 250 workstation, measured on a 120^3 lattice. For $\beta = 0.22311$ the integrated autocorrelation time of the magnetization was $\tau_{int} = 81(2)$ in units of sweeps.

4 Determination of Interface Free Energies

In order to obtain start values for the integration over β we simulated the 3D Ising model with fluctuating boundary conditions.

The β-values were chosen to be close to β_c, since here the interface free energies are small also on the larger lattices, and the boundary flip algorithm works still very well.

We chose eight different lattice volumes: The spatial extension (denoted by L in this paper) was 32, 48, 64, and 96. For the temporal extension (denoted by L_3) we used always two different values, namely $L_3 = L/2$ and $L_3 = L$. The data for the smaller extension in x_3-direction were taken in order to have a measure for the finite L_3 effects.

Our estimates together with a specification of the number of boundary flip updates to obtain them are given in table 1.

The next task then was to perform the numerical integration over the interface energies. Using the various algorithms described before, we obtained a large number of estimates for the interface energy, distributed on a fine grid of β-values, typically spaced by $\Delta\beta = 0.00005$ or 0.0001, see again table 1 for details. The numerical integration was done using the trapezoidal rule. For a check we also used interpolation with splines, yielding consistent results. Part of the integration was also done employing the "finite step $\Delta\beta$ method" described in [3].

After the laborious procedure we ended up (for each of the cross sections 32,48,64,96) with 71 estimates for the interface free energy, in the range $0.223 \leq \beta \leq 0.23$, spaced by $\Delta\beta = 0.0001$. For a selection of a few estimates, see our table 2). We decided not to quote our estimates for $\beta < 0.223$, since they seem to suffer from visible finite L_3 effects. Also, the data with $L_3 = L/2$, that were taken for $L = 32$ and $L = 96$ were after some inspection discarded from the further analysis.

5 Determination of the Interface Tension σ

In order to obtain estimates for the interface tensions, we employed eqs. (7) and (8).

We did five types of fits, to be labelled by **fit1**, **fit2**, **64vs48**, **96vs64**, and **fit3**. The various fit types are defined in the following table.

fit1	Fit the free energy data for L=32, 48, 64, and 96 with eq. (7)
fit2	Fit the free energy data for L= 48, 64, and 96 with eq. (7)
64vs48	Compute C_s and σ of eq. (7) from the $L = 64$ and 48 data alone
96vs64	Compute C_s and σ of eq. (7) from the $L = 96$ and 64 data alone
fit3	Fit the free energy data for L=32, 48, 64, and 96 with eq. (8)

These fits were applied to our free energy estimates ($L = 32, 48, 64, 96$) for the 71 different

β-values specified in the previous subsection. Our fit results for σ for a selection of β-values are given in table 3. With a few exceptions, the σ-estimates of the various fit types seem fairly consistent with each other, indicating that, for the range of β-values chosen, the effects from finite interface cross sections seem under control. Being cautious in estimating systematic errors, we have a tendency to announce the **96vs64** fit results as our final estimates for σ. The fits with the 2-loop approximation eq. (8) is, however, also very interesting. For $\beta = 0.223$ we did a more detailed analysis, comparing **fit1** and **fit3**, with further cross sections included. The results are summarized and explained in table 4. It seems that the fits with the 2-loop formula are more stable with respect to inclusion of smaller L-values, thus supporting the claims done in ref. [11].

In [22] it is suggested that the interface free energy should contain an extra term with a logarithmic L-dependence,

$$F_s = C_s + \sigma L^2 - \kappa \ln L + ... \tag{11}$$

with $\kappa \approx 1.65$. We note that our data are incompatible with such a big logarithmic correction.

It is interesting to look at the combination

$$G = C_s + \frac{1}{2} \ln \sigma \,. \tag{12}$$

This quantity has a scaling limit that can be compared with results of the semiclassical calculation of Münster [14]. This calculation yields

$$G = \ln 2 - \ln S \,, \tag{13}$$

with

$$S = 4 \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \left(1 - \frac{u_R}{4\pi}\left(\frac{39}{32} - \frac{15}{16}\ln 3\right)\right)^{-1/2} . \tag{14}$$

Assuming that $u_R = 14.3(1)$ in the critical limit [23], the semiclassical prediction (to this order) is

$$G \approx 0.29 \,. \tag{15}$$

Our numerical estimates for this quantity are given in table 5. Note that within the statistical errors there is a nice agreement of our results with the theoretical prediction.

6 Determination of σ_0

We fitted our estimates for σ with the ansatz

$$\sigma(\beta) = \sigma_0 \; t^{\mu} \; \left(1 \; + \; a_\theta \, t^{\theta} \; + \; a_1 \, t\right) . \tag{16}$$

In this ansatz, the leading corrections to scaling are included. Fits according to eq. (16) were recently applied by Zinn and Fisher [24] to our data published in [3].

We compared two different definitions of t, namely

$$t_1 \;\; = \;\; \beta/\beta_c - 1 \tag{17}$$

$$t_2 \;\; = \;\; 1 - \beta_c/\beta \,. \tag{18}$$

Agreement of fits with the two different definitions could be interpreted as a signal that effects from finite t (and/or higher order corrections) are under control. Note also that the coefficient a_1 should jump by μ when changing the definition of t.

In our fits, we always fixed the values of β_c, $\mu = 2\nu$, and θ. We consider the following choice of these parameters as reasonable: $\beta_c = 0.2216544(6)$ [9], $\mu = 2 \times 0.631$, $\theta = 0.51(3)$ [9].

To compare the different definitions of σ, cf. section 5, we did the fits for all the definitions. Always all 71 β-values were included in the fit. All the fits had a very good χ^2/dof (around one or smaller). Discarding the β-values farer away from criticality did not improve the quality of the fits.

Note that the σ-estimates for the various β-values are not completely statistically independent, since they stem from an integration procedure over the energy data. In order to take into account the cross-correlations properly, we redid all fits on a set of 50 suitably generated baby data sets for the σ's. The statistical estimate for the fit parameters was then obtained from the variance over the 50 babies.

In table 6 we give our results for the above quoted choice of the fixed parameters β_c, ν, and θ. One immediately recognizes that the estimates for the two different t-definitions agree nicely. However, the variation with the fit types seems a bit stronger. If we disregard the **64vs48** which does not include the $L = 96$ data, the results for σ_0 scatter from 1.54 to 1.57.

Of course, one has to check also the dependence of the fit results on the parameters fixed in the fit procedure. We found that the dependence on the choice of β_c and θ is quite weak, whereas the dependence on $\mu = 2\nu$ turns out to be non-negligible. We thus quote the results for one type of fit (**fit1**) for a range of "reasonable" ν-values in table 7. The estimate for σ_0 moves more or less from 1.51 to 1.59, whereas the relative variation of the other fit parameters is even stronger. Doing the same comparison with type **96vs64** yielded the same range of estimates for σ_0.

In summary, taking into account the systematic dependencies on the fit type and on the dependence on the input of the exponent ν we quote as our final estimate

$$\sigma_0 = 1.55(5)\,. \tag{19}$$

It is interesting to compare our result with those of the literature. We compiled a few of them in table 8. A fair agreement with most of the more recent estimates is found.

7 The Universal Amplitude Ratio R_-

Using the most recent numerical results for the second moment correlation length $\xi_{\rm 2nd}$ [23] we computed estimates for the universal amplitude ratio

$$R_- = \lim_{\beta \searrow \beta_c} \sigma(\beta)\, \xi_{\rm 2nd}(\beta)^2\,. \tag{20}$$

In order to extract the limiting value from our data we fitted according to

$$\sigma(\beta)\, \xi_{\rm 2nd}(\beta)^2 = R_- + c\, \xi^{-\omega}\,, \tag{21}$$

where we take the numerical value $\omega = 0.81(5)$ from the literature [9]. Including all β-values of table 9, we obtain $R_- = 0.1040(8)$ and $c = -0.023(1)$. The corresponding fit has a χ^2/dof

of 0.55. These results remain stable when the largest β-values are discarded. The error is dominated by the uncertainty of ω.

We again would like to compare our estimate with others in the literature. The perhaps most interesting comparison is with the semiclassical expansion of Münster [14]. It was recently extended to 2-loop [22]:

$$R_- = \frac{2}{u_R^*}\left(1 + \sigma_{1l}\frac{u_R^*}{4\pi} + \sigma_{2l}\left(\frac{u_R^*}{4\pi}\right)^2 + \ldots\right). \tag{22}$$

with $\sigma_{1l} = -.2002602$ and $\sigma_{2l} = -0.0076(8)$. Plugging in the most recent result for $u_R^* = 14.3(1)$ [23] one obtains for 1-loop $R_{-,1l} = 0.1080(10)$, and on 2-loop level $R_{-,2l} = 0.1066(10)$. In both cases the error is determined by the uncertainty of u_R^*.

It might be also interesting to compare with a few results obtained by numerical studies, see table 10.

Summary

We have presented a numerical study of the 3D Ising interface tension in the scaling region, using the method of "integration of the interface energy over β". Based on our results for the interface tension we estimated the amplitude σ_0 and the universal amplitude ratio R_-. Our results are $\sigma_0 = 1.55(5)$ and $R_- = 0.1040(8)$, respectively.

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List of Tables

β_0	L	L_3	*stat*	<boundary>	F_s	β-range	$\Delta\beta/10^{-4}$
0.2219	96	48	201,500	0.9056(10)	6.876(15)	0.2220-0.23	2/0.5
0.2219	96	96	201,000	0.8324(13)	6.956(8)	0.2220-0.23	2/0.5
0.2220	64	32	209,000	0.8560(12)	6.022(9)		
0.2220	64	64	213,300	0.7063(17)	5.918(7)	0.2221-0.23	1/0.5
0.2225	48	24	200,000	0.9306(10)	6.503(15)		
0.2225	48	48	200,000	0.8812(12)	6.633(11)	0.2227-0.23	0.5
0.2230	32	16	200,000	0.8999(11)	5.716(12)	0.2256-0.23	0.5
0.2230	32	32	225,000	0.8112(14)	5.727(8)	0.2230-0.23	0.5

Table 1: Interface free energy F_s for β close to β_c obtained from boundary flip simulations as described in ref. [5]. *stat* denotes the number of boundary flip updates, <boundary> is the expectation value of the boundary observable, which takes the value -1 for antiperiodic boundary and $+1$ for periodic boundary conditions. In the r.h.s. of the table we specify the range of β-values where Monte Carlo estimates for the interface energy were obtained. The β-values were spaced by increments $\Delta\beta$.

β	$L = 32$	$L = 48$	$L = 64$	$L = 96$
0.2230	5.727(08)	8.796(12)	13.088(27)	25.377(64)
0.2235	6.717(08)	11.295(12)	17.611(34)	35.879(71)
0.2240	7.844(08)	13.997(13)	22.535(37)	47.104(80)
0.2247	9.583(09)	18.051(14)	29.848(37)	63.725(85)
0.2255	11.718(09)	22.990(14)	38.748(38)	83.919(85)
0.2265	14.565(09)	29.544(15)	50.517(38)	110.563(86)
0.2275	17.567(10)	36.443(16)	62.870(39)	138.482(87)
0.2285	20.706(10)	43.620(17)	75.681(40)	167.441(88)
0.2300	25.610(11)	54.805(18)	95.639(41)	212.535(89)

Table 2: A selection of our interface free energy estimates computed in the range $0.223 \leq \beta \leq 0.23$.

fit	$\beta = 0.2230$	$\beta = 0.2240$	$\beta = 0.2255$	$\beta = 0.2275$	$\beta = 0.2300$
fit1	0.002398(5)	0.004793(6)	0.008808(6)	0.014753(9)	0.022810(8)
fit2	0.002398(6)	0.004784(9)	0.008810(9)	0.014759(11)	0.022813(11)
64vs48	0.002395(16)	0.004765(22)	0.008794(22)	0.014747(24)	0.022786(25)
96vs64	0.002400(14)	0.004799(17)	0.008822(18)	0.014768(19)	0.022831(19)
fit3	0.002376(5)	0.004782(6)	0.008801(6)	0.014750(9)	0.022808(8)

Table 3: Estimates for the interface tension σ as obtained from fitting the interface energies in various ways. The different types of fits are explained in the text.

L	F_s	χ^2/dof	C_s	σ	σ	C_s	χ^2/dof
8	2.669(09)	638.6	2.898(5)	0.002554(5)	0.002341(5)	3.681(6)	715.7
12	3.218(08)	240.7	3.053(5)	0.002486(5)	0.002346(5)	3.531(6)	42.9
18	3.912(11)	35.2	3.222(7)	0.002420(5)	0.002367(5)	3.458(8)	2.0
24	4.641(11)	7.1	3.288(9)	0.002399(6)	0.002369(6)	3.453(9)	2.2
30	5.459(13)	3.5	3.325(12)	0.002390(6)	0.002371(6)	3.446(12)	3.1
36	6.457(17)	2.5	3.362(21)	0.002382(7)	0.002369(7)	3.455(20)	5.8
64	13.088(27)		3.257(50)	0.002400(14)	0.002397(14)	3.293(50)	
96	25.377(64)						

Table 4: A comparison of two different fits to interface free energy data a $\beta = 0.223$. The second column gives the free energy estimates for the lattice cross sections ranging from $L = 8$ to $L = 96$. The data for $L = 64$ and $L = 96$ are taken from the present study, the estimates for the smaller lattices are taken from ref. [5]. Columns 3, 4 and 5 give the results of a fit with eq. (7). The first line gives the fit results when all data, starting from the $L = 8$ value, are included. The second lines gives the result when the $L = 8$ result is excluded from the data, the third line is based on discarding $L = 8$ and $L = 12$, and so on. In the last estimate only the two largest lattice sizes enter. In the right hand part of the table, we give for comparison the results for a fit with eq. (8).

fit	$\beta = 0.2230$	$\beta = 0.2240$	$\beta = 0.2255$	$\beta = 0.2275$	$\beta = 0.2300$
fit1	0.255(10)	0.269(11)	0.332(12)	0.349(16)	0.361(17)
fit2	0.253(21)	0.300(26)	0.323(29)	0.326(34)	0.348(32)
64vs48	0.260(44)	0.346(56)	0.363(58)	0.356(63)	0.415(67)
96vs64	0.241(70)	0.209(92)	0.247(96)	0.273(99)	0.231(103)
fit3	0.368(10)	0.328(11)	0.364(12)	0.368(16)	0.373(17)

Table 5: Estimates for the sums $G = C_s + \frac{1}{2}\ln\sigma$. The different types of fits are explained in the text.

fit	t-def	σ_0	a_θ	a_1
fit1	t_1	1.5677(73)	-0.563(47)	0.49(13)
	t_2	1.5682(75)	-0.561(49)	1.64(13)
fit2	t_1	1.549(11)	-0.409(72)	0.01(20)
	t_2	1.549(11)	-0.397(76)	1.13(21)
fit3	t_1	1.5428(73)	-0.376(49)	-0.06(14)
	t_2	1.5421(73)	-0.362(51)	1.06(14)
96vs64	t_1	1.571(20)	-0.57(13)	0.47(36)
	t_2	1.571(21)	-0.56(13)	1.61(37)
64vs48	t_1	1.519(23)	-0.18(16)	-0.68(44)
	t_2	1.517(23)	-0.15(16)	0.42(45)

Table 6: Fit results for the coefficients σ_0, a_θ, and a_1 in the critical law eq. (16). In these fits, we fixed the following parameters: $\beta_c = 0.2216544$, $\mu = 1.262$, and $\theta = 0.51$.

ν	σ_0	a_θ	a_1
0.628	1.5045(70)	-0.394(48)	0.17(13)
0.629	1.5253(71)	-0.451(48)	0.28(13)
0.630	1.5464(72)	-0.507(47)	0.38(13)
0.631	1.5677(73)	-0.563(47)	0.49(13)
0.632	1.5892(73)	-0.618(47)	0.59(13)

Table 7: Checking the dependence of the fit results for the coefficients σ_0, a_θ, and a_1 on the variation of $\nu = \mu/2$. The two other parameters, β_c and θ, are fixed to the values 0.2216544 and 0.51, respectively. The fit type is **fit1** here, and the t-type is t_1.

year	authors(s)	Ref.	σ_0
1982	Binder	[25]	1.05(5)
1984	Mon and Jasnow	[26]	1.2(1)
1988	Mon	[27]	1.58(5)
1992	Klessinger and Münster	[1]	1.29-1.64
1993	Berg et al.	[2]	1.52(5)
1993	Ito	[6]	1.42(4)
1993	Hasenbusch and Pinn	[3]	1.22-1.49
1993	Hasenbusch	[5]	1.5(1)
1993	Gausterer et al.	[4]	1.92(15)
1994	Caselle et al.	[11]	1.32-1.55
1996	Zinn and Fisher	[24]	1.50(1)
1997	Hasenbusch and Pinn	this work	1.55(5)

Table 8: Comparison of a number of estimates for σ_0 taken from the literature. The estimate by Zinn and Fisher is based on data from [3].

β	$\xi_{\rm 2nd}$	σ	$\sigma\,\xi^2_{\rm 2nd}$
0.2391	1.2335(15)	0.05555(10)	0.0845(3)
0.23142	1.8045(21)	0.02760(11)	0.0899(4)
0.2275	2.5114(31)	0.014768(19)	0.0931(3)
0.2260	3.0340(32)	0.010257(18)	0.0944(3)
0.2240	4.509(6)	0.004799(17)	0.0976(4)
0.22311	6.093(9)	0.002649(14)	0.0983(6)

Table 9: Results for the combination $\sigma\xi^2_{\rm 2nd}$. In the second column we give the results for the second moment correlation length obtained in ref. [23]. σ is our present estimate for the interface tension. In the last column we give $\sigma\xi^2_{\rm 2nd}$ obtained from the numerical results for $\xi_{\rm 2nd}$ and σ. The estimates are used to determine the universal amplitude ratio $R_- = \lim_{\beta\searrow\beta_c} \xi^2_{\rm 2nd}\,\sigma$.

year	authors(s)	Ref.	R_-
1992	Klessinger and Münster	[1]	0.090(3)
1993	Hasenbusch and Pinn	[3]	0.090(5)
1996	Zinn and Fisher	[24]	0.096(2)
1996	Agostini et al.	[28]	0.1056(19)
1997	Hasenbusch and Pinn	this work	0.1040(8)

Table 10: Comparison of a number of estimates for R_- taken from the literature. The estimate of Zinn and Fisher is based on data of [3]. Agostini et al. used the true instead of the second moment correlation length.